

Addendum to Pentapods with Mobility 2

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In a foregoing publication the authors studied pentapods with mobility 2, where neither all platform anchor points nor all base anchor points are located on a line. It turned out that the given classification is incomplete. This addendum is devoted to the discussion of the missing cases resulting in additional solutions already known to Duporcq.

1 Introduction

The geometry of a pentapod is given by the five base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, C_i)^T$ with respect to the fixed system and by the five platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, b_i, c_i)^T$ with respect to the moving system (for $i = 1, \dots, 5$). Each pair (M_i, m_i) of corresponding anchor points is connected by a SPS-leg, where only the prismatic joint (P) is active and the spherical joints (S) are passive.

If the geometry of the manipulator is given, as well as the lengths of the five pairwise distinct legs, the pentapod has generically mobility 1 according to the formula of Grübler. The corresponding motion is called a 1-dimensional self-motion of the pentapod. But, under particular conditions, the manipulator can gain additional mobility. We can focus on pentapods with mobility 2, as those with higher-dimensional self-motions are already known (cf. [1, Corollary 1]).

1.1 Reason for the Addendum

The classification of pentapods with mobility 2 given in [1] was based on the following theorem of [2]:

Theorem 1. *If the mobility of a pentapod is 2 or higher, then one of the following conditions holds¹:*

¹After a possible necessary renumbering of anchor points and exchange of the platform and the base.

- (a) The platform and the base are similar. This is a so-called equiform pentapod.
- (b) The platform and the base are planar and affine equivalent. This is a so-called planar affine pentapod.
- (c) There exists $p \leq 5$ such that m_1, \dots, m_p are collinear and M_{p+1}, \dots, M_5 are equal; i.e., $M_{p+1} = \dots = M_5$.
- (d) M_1, M_2, M_3 are located on the line g which is parallel to the line h spanned by M_4 and M_5 . Moreover m_1, m_2, m_3 are located on the line g' which is parallel to the line h' spanned by m_4 and m_5 .

During the literature research for the article [3, Section 1], we came across the work [4] of Duporcq, which describes the following remarkable motion (see Fig. 1):

Let M_1, \dots, M_6 and m_1, \dots, m_6 be the vertices of two complete quadrilaterals, which are congruent. Moreover the vertices are labeled in a way that m_i is the opposite vertex of M_i for $i \in \{1, \dots, 6\}$. Then there exist a two-parametric line-symmetric motion where each m_i is running on spheres centered in M_i .

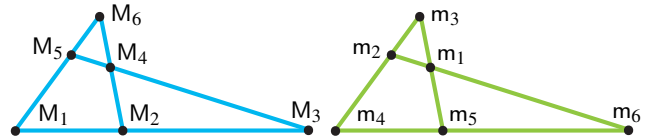


Fig. 1. Illustration of Duporcq's complete quadrilaterals.

It can easily be checked that this configuration of anchor points corresponds to an architecturally singular hexapod (e.g. [5] or [6]). As architecturally singular manipulators are redundant we can remove any leg — without loss of generality (w.l.o.g.) we suppose that this is the sixth leg — without changing the direct kinematics of the mechanism. Therefore

the resulting pentapod M_1, \dots, M_5 and m_1, \dots, m_5 , which we call a *Duporcq pentapod* for short, has also a two-parametric line-symmetric self-motion. This yields a counter-example to Theorem 1, but the flaw can be fixed by adding the following case to Theorem 1 (cf. [7]):

(e) *The following triples of points are collinear:*

$$M_1, M_2, M_3, \quad M_3, M_4, M_5, \quad m_3, m_1, m_i, \quad m_3, m_j, m_k,$$

with pairwise distinct $i, j, k \in \{2, 4, 5\}$. Moreover the points M_1, \dots, M_5 are pairwise distinct as well as the points m_1, \dots, m_5 .

As Theorem 1 only gives necessary conditions, the addendum is devoted to the determination of sufficient ones for the 2-dimensional mobility of pentapods belonging to item (e). In detail the paper is structured as follows:

In Section 2 further necessary conditions are obtained by means of Möbius photogrammetry, which restrict the pentapod designs of (e) to three possible cases, up to affinities of the planar platform and the planar base. In Section 3 we repeat the theory of bonds based on two different embeddings of $SE(3)$ and prove Lemmata 1 and 2 as well as Corollaries 1 and 2. Based on these results we show in Section 4 that only the Duporcq pentapods were missed by our classification given in [1]. The consequence of this result for article [1] are summed up in the conclusions (Section 5).

2 Möbius photogrammetric considerations

In Subsection 2.1 we recall some basics of Möbius photogrammetry, which are needed for the construction of the three possible pentapod designs (up to affinities of the planar platform and the planar base) given in Subsection 2.2.

2.1 Basics

First of all we need the notation of a so-called Möbius transformation γ of the plane. If we combine the planar Cartesian coordinates (x, y) to a complex number $z := x + iy$, then $\gamma(z)$ can be defined as a rational function of the form

$$\gamma: z \mapsto \frac{z_1 z + z_2}{z_3 z + z_4}, \quad (1)$$

with complex numbers z_1, \dots, z_4 satisfying $z_1 z_4 - z_2 z_3 \neq 0$. Therefore Möbius transformations can be seen as the projective transformations of the complex projective line $\mathbb{P}_{\mathbb{C}}^1$.

We identify by the mapping ι the unit-sphere S^2 of the Euclidean 3-space \mathbb{R}^3 with an algebraic curve $C := \{x^2 + y^2 + z^2 = 0\}$ in $\mathbb{P}_{\mathbb{C}}^2$. In detail this identification works as follows: Let $\mathbf{u} \in S^2$ with $\mathbf{u} = (u_1, u_2, u_3)$. Then determine $\mathbf{v}, \mathbf{w} \in S^2$ with $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ in a way that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ determine a right-handed basis of \mathbb{R}^3 . Then the map $\iota: S^2 \rightarrow \mathbb{P}_{\mathbb{C}}^2$ is given by:

$$\iota: (u_1, u_2, u_3) \mapsto (v_1 + iw_1 : v_2 + iw_2 : v_3 + iw_3) \quad (2)$$

as a different choice of $\mathbf{v}, \mathbf{w} \in S^2$ yields to the same point in $\mathbb{P}_{\mathbb{C}}^2$.

By denoting the vector (M_1, \dots, M_5) of five points in \mathbb{R}^3 by \mathfrak{M} , the orthogonal parallel projection π of \mathfrak{M} along the direction associated with $c \in C$ is given by $\pi_c(\mathfrak{M})$. By writing the planar Cartesian coordinates of each projected point as a complex number we get $\pi_c(\mathfrak{M}) \in (\mathbb{P}_{\mathbb{C}}^1)^5$.

Remark 1. Assume that M_1, \dots, M_5 is known to be coplanar, the 5-tuple can be reconstructed from $\pi_c(\mathfrak{M})$ only up to affinity, as the orientation of the carrier plane of the 5 points with respect to $\iota^{-1}(c)$ is not known. This also corrects [7], where "similarity" is written instead of "affinity". \diamond

The equivalence class under the action of the Möbius group Γ on $\pi_c(\mathfrak{M})$ is the so-called Möbius picture $[\pi_c(\mathfrak{M})]_{\Gamma}$ of \mathfrak{M} along the direction associated with $c \in C$.

The set of all these equivalence classes $[(\mathbb{P}_{\mathbb{C}}^1)^5]_{\Gamma}$ can be viewed as a quintic surface $P_5 \in \mathbb{P}_{\mathbb{C}}^5$ known as Del Pezzo surface. For $\pi_c(\mathfrak{M})$ with coordinates $(x_1 + iy_1, \dots, x_5 + iy_5) \in (\mathbb{P}_{\mathbb{C}}^1)^5$ the corresponding point of the Del Pezzo surface is defined as $(\varphi_0 : \varphi_1 : \varphi_2 : \varphi_3 : \varphi_4 : \varphi_5)$ with:

$$\begin{aligned} \varphi_0 &:= D_{12}D_{23}D_{34}D_{45}D_{15}, \\ \varphi_1 &:= D_{12}D_{25}D_{15}D_{34}D_{34}, \\ \varphi_2 &:= D_{12}D_{23}D_{13}D_{45}D_{45}, \\ \varphi_3 &:= D_{23}D_{34}D_{24}D_{15}D_{15}, \\ \varphi_4 &:= D_{34}D_{45}D_{35}D_{12}D_{12}, \\ \varphi_5 &:= D_{14}D_{45}D_{15}D_{23}D_{23}, \end{aligned} \quad (3)$$

and $D_{ij} := x_i y_j - x_j y_i$. For details of this construction of P_5 we refer to [2, Section 3.1], but it is important to note that P_5 carries 10 lines L_{ij} corresponding to equivalence classes for which the projection of the i th and the j th point coincide ($\Leftrightarrow D_{ij} = 0$) for pairwise distinct $i, j \in \{1, \dots, 5\}$.

We are interested in the set of Möbius pictures of \mathfrak{M} under all $c \in C$. By applying the so-called photographic map $f_{\mathfrak{M}}$ of \mathfrak{M} given by

$$f_{\mathfrak{M}}: C \rightarrow P_5 \quad \text{with} \quad c \mapsto [\pi_c(\mathfrak{M})]_{\Gamma} \quad (4)$$

we can compute the so-called profile $p_{\mathfrak{M}}$ of \mathfrak{M} as the Zariski closure of $f_{\mathfrak{M}}(C)$; i.e. $\text{ZarClo}(f_{\mathfrak{M}}(C))$. Note that the profile is a curve on P_5 .

According to [2, Remark 3.5] the Möbius picture cannot be defined for those values $c \in C$, for which the associated directions are parallel to three collinear points of \mathfrak{M} , as in this case all five φ_i 's are equal to zero. In our case two such directions exist, which are parallel to the carrier line of M_1, M_2, M_3 (i.e. the metallic direction m) and M_3, M_4, M_5 (i.e. the blue direction b), respectively (cf. Fig. 2). However, we can extend $f_{\mathfrak{M}}$ also to these directions by canceling out the common vanishing factor. For our given base (cf. Fig. 2) this common factor is $D_{12} = D_{13} = D_{23}$ (resp. $D_{34} = D_{35} = D_{45}$) for the metallic (resp. blue) direction, thus Eq. (3) yields $(0 : D_{25}D_{34} : 0 : D_{24}D_{15} : 0 : 0)$ (resp.

$(0 : 0 : 0 : D_{24}D_{15} : 0 : D_{14}D_{23}))$. Therefore the m -direction (resp. b -direction) is mapped on a point of L_{45} (resp. L_{12}); i.e.

$$[\pi_m(\mathfrak{M})]_\Gamma \in L_{45}, \quad [\pi_b(\mathfrak{M})]_\Gamma \in L_{12}. \quad (5)$$

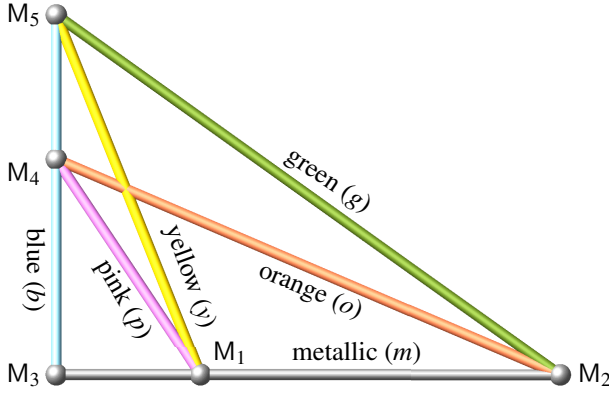


Fig. 2. The photographic map sends any direction vector parallel to a line through 2 (but not 3) of the points M_i, M_j to the unique point in the Möbius picture on the line L_{ij} of the quintic surface P_5 . In the base configuration above, green (g) is sent to L_{25} , orange (o) is sent to L_{24} , yellow (y) is sent to L_{15} and pink (p) is sent to L_{14} . It is not clear whether the directions blue (b) and metallic (m) are being sent; later, we will show that b is sent to L_{12} and m is sent to L_{45} .

2.2 Three possible designs

It is known (see [2, Section 4]) that for a pentapod with mobility 2, which belongs to item (e) of Theorem 1, the profiles $p_{\mathfrak{M}}$ and p_m have to coincide, where m denotes the vector of five points (m_1, \dots, m_5) . As a consequence there has to be a one-to-one correspondence between $p_{\mathfrak{M}}$ and p_m , which is used to reconstruct in three ways m (up to affinity; cf. Remark 1), under the assumption that \mathfrak{M} is given (cf. Fig. 2).

Assumption 1. *W.l.o.g. we can assume that the reconstruction m is affinely transformed in a way that the Möbius pictures (and their extension in the case of three collinear points) of m and \mathfrak{M} with respect to any direction c are identical.*

First of all we have to distinguish the following three cases, which are implied by the three possible collinearity configurations stated in (e):

1. $i = 2$: W.l.o.g. we can set $j = 4$ and $k = 5$.

One can select m_1 arbitrarily. As $L_{14} \cap p_m$ has to coincide with $L_{14} \cap p_{\mathfrak{M}}$ the line $m_1 m_4$ has to be parallel to $M_1 M_4$. Now we can select any point ($\neq m_1$) on the parallel line to $M_1 M_4$ through m_1 as m_4 . The direction of $m_1 m_2$ is not uniquely determined as the line $M_1 M_2$ also

contains the point M_3 . Due to the one-to-one correspondence between the two profiles, $L_{12} \cap p_m$ has to correspond with one of the two points on $p_{\mathfrak{M}}$, which do not admit a Möbius picture. Therefore there are the following two possibilities:

- (a) $m_1 m_2$ is parallel to $M_1 M_2$: As a consequence $m_4 m_5$ has to be parallel to $M_4 M_5$.

Moreover as $L_{24} \cap p_m$ has to coincide with $L_{24} \cap p_{\mathfrak{M}}$ the line $m_2 m_4$ has to be parallel to $M_2 M_4$. Therefore we get m_2 as the intersection point of a parallel line to $M_1 M_2$ through m_1 and a parallel line to $M_2 M_4$ through m_4 .

In the same way $L_{15} \cap p_m$ has to coincide with $L_{15} \cap p_{\mathfrak{M}}$ and therefore m_5 can be obtained as the intersection point of a parallel line to $M_4 M_5$ through m_4 and a parallel line to $M_1 M_5$ through m_1 .

Although all points are reconstructed, we have to check if the last remaining condition is fulfilled, namely if $m_2 m_5$ is parallel to $M_2 M_5$. As this can easily be verified, we get reconstruction 1 illustrated in Fig. 3(left), which is in fact identical with \mathfrak{M} (cf. Fig. 2).

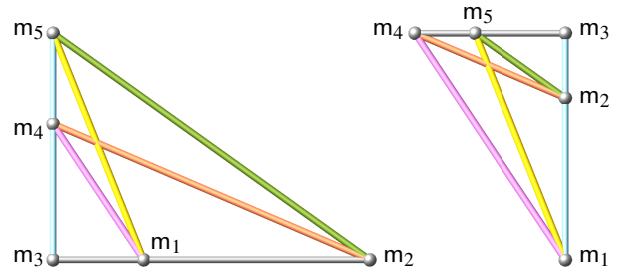


Fig. 3. The two possible reconstructions of the platform configuration from the Möbius picture, under the additional assumption that m_3 is the intersection of lines $m_1 m_2$ and $m_3 m_4$. Note that the line $m_2 m_5$ must have direction g , the line $m_2 m_4$ must have direction o , the line $m_1 m_5$ must have direction y and the line $m_1 m_4$ must have direction p . The left configuration coincides with the base configuration. We will see later that the right configuration is not compatible because the lines through m_1, m_2, m_3 and m_3, m_4, m_5 , respectively, do not have the correct directions.

- (b) $m_1 m_2$ is parallel to $M_4 M_5$: As a consequence $m_4 m_5$ has to be parallel to $M_1 M_2$.

Analogous arguments as in the above case with respect to the swapped directions yield a further candidate platform m illustrated in Fig. 3(right). Calculation of the Möbius picture of m with respect to the directions m and b according to Eq. (3) shows

$$[\pi_m(m)]_\Gamma \in L_{12}, \quad [\pi_b(m)]_\Gamma \in L_{45}. \quad (6)$$

Due to Eq. (5), m and \mathfrak{M} do not have the same Möbius picture with respect to the directions m and b ; a contradiction.

2. $i = 5$: W.l.o.g. we can set $j = 2$ and $k = 4$.

For the same reasons as in item 1 we can select m_1 arbitrarily and can choose any point ($\neq m_1$) on the parallel line to M_1M_4 through m_1 as m_4 . Moreover the following two subcases can also be reasoned analogously to item 1:

(a) m_1m_2 is parallel to M_1M_2 : As a consequence m_4m_5 has to be parallel to M_4M_5 .

In this case we also get Eq. (6), which implies the same contradiction as in case 1(b).

(b) m_1m_2 is parallel to M_4M_5 : As a consequence m_4m_5 has to be parallel to M_1M_2 .

As now the line m_2m_4 also contains the point m_3 the corresponding direction does not admit a Möbius picture. Due to the one-to-one correspondence between $p_{\mathcal{M}}$ and p_m again two cases have to be distinguished:

i. m_2m_4 is parallel to M_1M_5 : As a consequence m_1m_5 has to be parallel to M_2M_4 .

Therefore m_2 can be obtained as the intersection point of the parallel line to M_1M_2 through m_1 and the parallel line to M_1M_5 through m_4 . Moreover m_5 equals the intersection point of the parallel line to M_4M_5 through m_4 and the parallel line to M_2M_4 through m_1 .

Although all points are reconstructed, we have to check again if the last remaining condition is fulfilled, namely if m_2m_5 is parallel to M_2M_5 . As this can easily be verified, we get reconstruction 2 illustrated in Fig. 4(right).

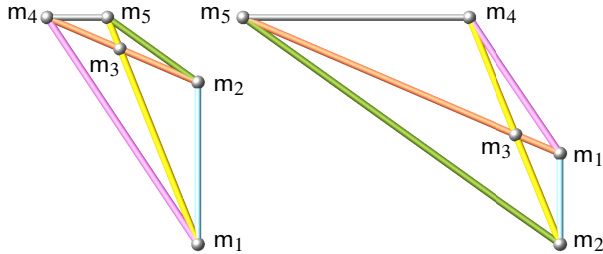


Fig. 4. The two possible reconstructions of the platform configuration from the Möbius picture, under the additional assumption that m_3 is the intersection of lines m_1m_5 and m_2m_4 . Here the directions of lines m_1m_2 , m_4m_5 , m_1m_4 and m_2m_5 are fixed to b , m , p and g , respectively. We will later see that the left configuration is not compatible. The right configuration leads to a Duporcq pentapod.

ii. m_2m_4 is parallel to M_2M_4 : As a consequence m_1m_5 has to be parallel to M_1M_5 .

Analogous considerations as in item 2(b)i yields the candidate platform illustrated in Fig. 4(left). Now the calculation of the Möbius picture of this candidate with respect to the orange direction o yields $(0 : 0 : D_{13}D_{45} : 0 : D_{35}D_{12} : 0)$. Therefore we have $[\pi_o(m)]_\Gamma \in L_{15}$, which contradicts

$[\pi_o(m)]_\Gamma \in L_{24}$, thus we have no valid reconstruction.

3. $i = 4$: W.l.o.g. we can set $j = 2$ and $k = 5$.

The discussion of cases is exactly the same as in item 2 if one exchanges the indices 4 and 5. The resulting reconstruction 3 as well as the corresponding non-valid candidate platform are illustrated in Fig. 5.

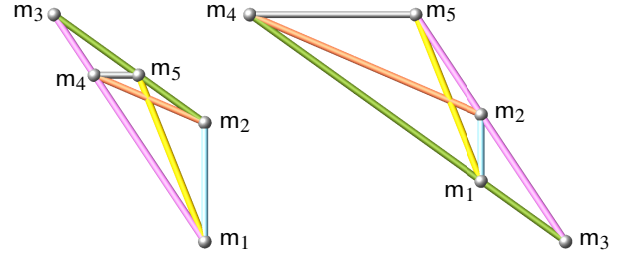


Fig. 5. The two possible reconstructions of the platform configuration from the Möbius picture, under the additional assumption that m_3 is the intersection of lines m_1m_4 and m_2m_5 . Here the directions of lines m_1m_2 , m_4m_5 , m_1m_5 and m_2m_4 are fixed to b , m , y , and o , respectively. The left configuration is not compatible. The right configuration leads to a Duporcq pentapod.

Moreover for the i th reconstruction ($i = 1, 2, 3$) there exists an affine relation κ_i between the set $\{M_1, M_2, M_4, M_5\}$ and the set $\{m_1, m_2, m_4, m_5\}$. In detail these affine mappings κ_i are given by:

$$\begin{aligned} \kappa_1 : & M_1 \mapsto m_1 & M_2 \mapsto m_2 & M_4 \mapsto m_4 & M_5 \mapsto m_5 \\ \kappa_2 : & M_1 \mapsto m_4 & M_2 \mapsto m_5 & M_4 \mapsto m_1 & M_5 \mapsto m_2 \\ \kappa_3 : & M_1 \mapsto m_5 & M_2 \mapsto m_4 & M_4 \mapsto m_2 & M_5 \mapsto m_1 \end{aligned}$$

For all three cases the validity of these affine mappings can be proven by direct computation. Moreover it should be noted that κ_1 maps $M_3 \mapsto m_3$ in addition. As a consequence the pentapod design resulting from reconstruction 1 is a planar affine pentapod belonging to item (b) of Theorem 1, which was already discussed in [1]. Therefore we remain only with reconstruction 2 and 3.

Remark 2. Note that the Duporcq pentapods fit with reconstruction 2 and 3 for the following reason: Assumed the base (cf. Fig. 2) is given, then two lines of the complete quadrilateral through the points M_1, M_2, M_4, M_5 are already determined by the collinearity of the triples M_1, M_2, M_3 and M_3, M_4, M_5 , respectively. Therefore the quadrilateral is completed either by the lines M_1M_5 and M_2M_4 , which corresponds with reconstruction 2, or by the lines M_1M_4 and M_2M_5 , which corresponds with reconstruction 3. \diamond

3 Bond Theory

In this section we shortly repeat two different approaches for defining so-called bonds. The first one dis-

cussed in Subsection 3.1 is based on the Study parametrization of $SE(3)$ in contrast to the one presented in Subsection 3.2, which uses the so-called conformal embedding of $SE(3)$. In Section 3.3 a relation between the bonds based on these different embeddings is given.

3.1 Bonds based on the Study Embedding of $SE(3)$

We denote the eight homogenous Study parameters by $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$, where the first four homogeneous coordinates $(e_0 : e_1 : e_2 : e_3)$ are the so-called Euler parameters. Now, all real points of the Study parameter space \mathbb{P}^7 , which are located on the so-called Study quadric $S : \sum_{i=0}^3 e_i f_i = 0$, correspond to an Euclidean displacement, with exception of the 3-dimensional subspace $e_0 = e_1 = e_2 = e_3 = 0$, as its points cannot fulfill the condition $N \neq 0$ with $N = e_0^2 + e_1^2 + e_2^2 + e_3^2$. All points of the complex extension $\mathbb{P}_{\mathbb{C}}^7$ of \mathbb{P}^7 , which cannot fulfill this normalizing condition, are located on the so-called exceptional quadric $N = 0$.

By using the Study parametrization of Euclidean displacements, the condition that the point m_i is located on a sphere centered in M_i with radius R_i is a quadratic homogeneous equation according to Husty [8]. For the explicit formula of this so-called sphere condition Q_i we refer to [1, Eq. (2)]. Now the solution for the direct kinematics over \mathbb{C} of a pentapod can be written as the algebraic variety V of the ideal spanned by $S, Q_1, \dots, Q_5, N = 1$. In the case of pentapods with mobility 2 the variety V is 2-dimensional.

We consider the algebraic motion of the pentapod, which is defined as the set of points on the Study quadric determined by the constraints; i.e., the common points of the six quadrics S, Q_1, \dots, Q_5 . Now the points of the algebraic motion with $N \neq 0$ equal the kinematic image of the algebraic variety V . But we can also consider the set \mathcal{B} of points of the algebraic motion, which belong to the exceptional quadric $N = 0$. For an exact mathematical definition of these so-called bonds we refer to [1, Definition 1]. In the case of pentapods with mobility 2 the set \mathcal{B} is of dimension 1; i.e., a *bonding curve*.

We use the following approach for the computation of bonds: In a first step we project the algebraic motion of the pentapod into the Euler parameter space $\mathbb{P}_{\mathbb{C}}^3$ by the elimination of f_0, \dots, f_3 . This projection is denoted by ζ . In a second step we determine the set \mathcal{B}_{ζ} of projected bonds as those points of the projected point set $\zeta(V)$, which are located on the quadric $N = 0$; i.e.,

$$\mathcal{B}_{\zeta} := \text{ZarClo}(\zeta(V)) \cap \{(e_0 : \dots : e_3) \in \mathbb{P}_{\mathbb{C}}^3 \mid N = 0\}. \quad (7)$$

3.2 Bonds based on the Conformal Embedding of $SE(3)$

As shown in [9, Section 2.1], it is possible to construct a projective compactification X in $\mathbb{P}_{\mathbb{C}}^{16}$ for the complexification $SE(3)_{\mathbb{C}}$ of the group $SE(3)$ in a way that the sphere condition is linear in the coordinates of $\mathbb{P}_{\mathbb{C}}^{16}$. The map $SE(3) \hookrightarrow \mathbb{P}_{\mathbb{C}}^{16}$ is the so-called *conformal embedding* of $SE(3)$ and X is a projective variety of dimension 6 and degree 40.

Now the five linear sphere conditions determine a linear subspace $F \subseteq \mathbb{P}_{\mathbb{C}}^{16}$ of codimension 5. The intersection $K = X \cap F$ is defined to be the *complex configuration set* of the pentapod.

It is also known that X can be written as the disjoint union $SE(3)_{\mathbb{C}} \cup B_X$, where the so-called boundary B_X is obtained as the intersection of X and a hyperplane H . Moreover the boundary can be decomposed into the following 5 subsets:

Vertex: This is the only real point in B_X , a singular point with multiplicity 20; it is never contained in K .

Collinearity points: If K contains such a point, then either the platform points or the base points are collinear.

Similarity points: If K contains such a point, then there are normal projections of platform and base to a plane such that the images are similar.

Inversion points: If K contains such a point, then there are normal projections of platform and base to a plane such that the images are related by an inversion.

Butterfly points: If K contains such a point, then there are two lines, one in the base and one in the platform, such that any leg has either its base point on the base line or its platform point on the platform line.

Now the set of bonds \mathcal{B}_K is obtained as the intersection of K and the boundary B_X . Moreover it should be mentioned that the intersection multiplicity of K and H is at least 2 in each bond. Note that for pentapods with mobility 2, the bondset \mathcal{B}_K is 1-dimensional.

3.3 Relation between Bonds based on different Embeddings

If $\rho : SE(3) \rightarrow SO(3)$ is the map sending a direct isometry to its rotational part, then there exists a linear projection $\xi : \mathbb{P}_{\mathbb{C}}^{16} \dashrightarrow \mathbb{P}_{\mathbb{C}}^9$ such that the following diagram is commutative (cf. [10, Section 1]):

$$\begin{array}{ccc} SE(3) & \xrightarrow{\quad} & X \subseteq \mathbb{P}_{\mathbb{C}}^{16} \\ \rho \downarrow & \searrow & \downarrow \xi \\ SO(3) & \xrightarrow{\quad} & \mathbb{P}_{\mathbb{C}}^3 \xrightarrow{v_{3,2}} V_{3,2} \subseteq \mathbb{P}_{\mathbb{C}}^9 \end{array} \quad (8)$$

$\begin{array}{c} S \subseteq \mathbb{P}_{\mathbb{C}}^7 \\ \downarrow \zeta \\ \mathbb{P}_{\mathbb{C}}^3 \end{array}$

where $v_{3,2}$ is the Veronese embedding of $\mathbb{P}_{\mathbb{C}}^3$ and $V_{3,2}$ is its image in $\mathbb{P}_{\mathbb{C}}^9$. The center of ξ is the linear space spanned by similarity points, which contains also the collinearity points and the vertex.

Lemma 1. *For reconstruction 2 and 3 the complex configuration set K does not contain collinearity bonds, but four butterfly bonds and one similarity bond.*

Proof. The numbers of collinearity and butterfly bonds are trivial. The reasoning for the existence of exactly one similarity bond is as follows (cf. Fig. 6):

As M_1, M_2, M_3 are collinear the ratio $TV(M_1, M_2, M_3)$ remains constant under parallel projections (with projection directions not parallel to the carrier line of the collinear points). Therefore one can construct the point m'_3 on the line $m_1 m_2$ such that $TV(m_1, m_2, m'_3) = TV(M_1, M_2, M_3)$ holds. In the same way one can construct the point m''_3 on the line $m_4 m_5$ such that $TV(m'_3, m_4, m_5) = TV(M_3, M_4, M_5)$ holds. It can be checked by direct computations that m_3, m'_3, m''_3 are located on a line g , which gives the direction of the projection of the platform.

The reverse construction from the platform to the base yields the points M_3, M'_3, M''_3 located on a line G , which gives the direction of the projection of the base.

Moreover g and G have to be parallel due to Assumption 1, which can also be checked by straightforward computations. \square

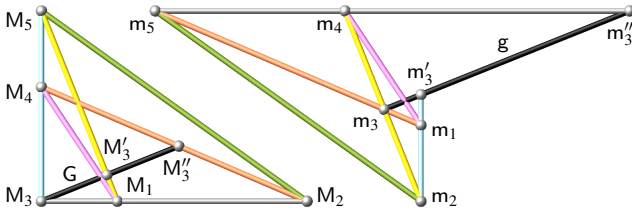


Fig. 6. Projection along the black direction leads to one-dimensional configurations that are similar. This shows that the pentapod with such base/platform configuration has a similarity bond.

Corollary 1. *If reconstruction 2 or 3 has mobility 2, then K_e has to be a surface.*

Proof. First of all we want to recall the known characterization for a pure translational self-motion (according to [11, Theorem 2] under consideration of [1, Footnote 4]): A pentapod possesses a pure translational self-motion, if and only if the platform can be rotated about the center $m_1 = M_1$ into a pose, where the vectors $\overrightarrow{M_i m_i}$ for $i = 2, \dots, 5$ fulfill the condition $rk(\overrightarrow{M_2 m_2}, \dots, \overrightarrow{M_5 m_5}) \leq 1$. Note that this implies the existence of a similarity bond. Moreover all 1-dimensional self-motions are circular translations in planes orthogonal to the parallel vectors $\overrightarrow{M_i m_i}$ for $i = 2, \dots, 5$.

If K_e is a point, then the orientation during the self-motion is fixed and we can only obtain a 2-dimensional translational self-motion. It is well-known [11] that in this case the platform and the base have to be directly congruent.

If K_e is a curve, then for each corresponding platform orientation a 1-dimensional translational sub-self-motion has to exist. This implies a 1-dimensional set of similarity bonds, which contradicts Lemma 1. \square

Lemma 2. *Assume that K is a surface. Assume that the projection $v_{3,2}^{-1} \circ \xi: K \mapsto K_e$ is birational. Assume that the pod has infinitely many inversion bonds. Then the intersection of K_e and the exceptional quadric $N = 0$ has a curve of multiplicity bigger than 1.*

Proof. The mobility surface K has a tangential intersection with the boundary hyperplane H at all inversion bonds. Since the projection $K \rightarrow K_e$ is birational, the projection $\xi: X \subset \mathbb{P}^{16} \dashrightarrow \mathbb{P}^9$ is locally an isomorphism for almost all inversion points. Hence the image of K intersects the image of the hyperplane H tangentially at almost all images of inversion points. This is equivalent to saying that K_e intersects the exceptional quadric $N = 0$ at almost all images of inversion points. The closure of these points would be the curve with intersection multiplicity bigger than 1. \square

Corollary 2. *If in addition to the above assumptions K_e is a quadric surface, then the intersection is totally tangential along an irreducible quadric.*

Proof. By degree, the part D of the intersection which has multiplicity bigger than one can only be a line or a conic. Since D is defined over \mathbb{R} , and any line contains real points, and the exceptional quadric does not have real points, D is not a line. If D were reducible, then it is the union of two lines. If the two lines intersect, then the intersection point would be real, but the exceptional quadric $N = 0$ contains no real points; a contradiction. If the two lines do not intersect, then we have a complete intersection of dimension 1 which is disconnected, and this is also in contradiction to a well-known theorem in algebraic geometry (see [12]). Therefore we remain with the case stated in Corollary 2. \square

4 Computations in Study and Euler parameter space

Within this section we prove computationally that reconstruction 2 and 3 can only have a 2-dimensional self-motion in the case already known to Duporcq.

4.1 Computation of K_e

We parametrize the base as follows:

$$\begin{aligned} \mathbf{M}_1 &:= (0, 0, 0)^T, & \mathbf{M}_2 &:= (1, 0, 0)^T, \\ \mathbf{M}_4 &:= (A_4, B_4, 0)^T, & \mathbf{M}_5 &:= (A_5, B_5, 0)^T. \end{aligned} \quad (9)$$

Then M_3 is already determined as the intersection point of $M_1 M_2$ and $M_4 M_5$, thus we get:

$$\mathbf{M}_3 := \left(\frac{B_4 A_5 - A_4 B_5}{B_4 - B_5}, 0, 0 \right)^T. \quad (10)$$

As M_3 has to be a finite point which is not allowed to collapse with one of the other four given points we have $B_4 B_5 U_1 \neq 0$ with:

$$U_1 := (B_4 - B_5)(B_4 A_5 - A_4 B_5)(B_4 A_5 - A_4 B_5 - B_4 + B_5). \quad (11)$$

Now we compute the platform with the help of the mapping κ_i . In the remainder of this section we restrict to the case

$i = 2$ as the case $i = 3$ can be done in a total analogous way. W.l.o.g. we can assume that the matrix \mathbf{A} of κ_2 has the form:

$$\mathbf{A} := \begin{pmatrix} \mu_1 & \mu_2 \\ 0 & \mu_3 \end{pmatrix} \quad \text{with} \quad \mu_1\mu_3 \neq 0 \quad \text{and} \quad \mu_1 > 0. \quad (12)$$

Therefore we get $\mathbf{m}_j = \mathbf{A}\mathbf{M}_j$ for $j = 1, 2, 4, 5$ and obtain \mathbf{m}_3 as the intersection point of $\mathbf{m}_2\mathbf{m}_4$ and $\mathbf{m}_1\mathbf{m}_5$, which yields:

$$\mathbf{m}_3 := \left(\frac{B_4(A_5\mu_1 + B_5\mu_2)}{B_4A_5 + B_5 - A_4B_5}, \frac{B_4B_5\mu_3}{B_4A_5 + B_5 - A_4B_5}, 0 \right)^T. \quad (13)$$

As this point also has to be a finite point we get additionally the assumption $U_2 \neq 0$ with

$$U_2 := B_4A_5 + B_5 - A_4B_5. \quad (14)$$

Moreover \mathbf{M}_3 cannot be located on $\mathbf{M}_4\mathbf{M}_5$ which yields $U_3 \neq 0$ with

$$U_3 := B_4A_5 - B_4 - A_4B_5. \quad (15)$$

Let us denote the numerator of the difference $Q_1 - Q_i$ ($i = 2, \dots, 5$) of sphere conditions by Δ_i . Then we can compute the linear combination

$$B_4B_5U_1\Delta_2 + U_3\Delta_3 + B_5U_1U_2\Delta_4 - B_4U_1U_2\Delta_5 \quad (16)$$

which yields a quadratic expression $K_e[1356]$ in the Euler parameters (free of Study parameters f_0, \dots, f_3), where the number in the bracket gives the number of terms.

Remark 3. K_e cannot be fulfilled identically for the following reason: In this case the sphere conditions Q_1, \dots, Q_5 are linearly dependent and therefore we would end up with an degenerated architectural singular manipulator [13], which has to have 4 collinear points in the platform or the base (see also [14]) contradicting the design under consideration. \diamond

4.2 Determining the pentapod's geometry

In the following we show that the projection $K \rightarrow K_e$ cannot be birational. This is done by contradiction; i.e. we assume that $K \rightarrow K_e$ is birational, and show that K_e cannot intersect $N = 0$ totally tangential along an irreducible quadric (cf. Corollary 2).

If K_e touches $N = 0$ along a quadric then there has to exist a double-counted plane \mathcal{E} : $v_0e_0 + v_1e_1 + v_2e_2 + v_3e_3 = 0$ within the pencil of quadrics spanned by $K_e = 0$ and $N = 0$. Thus we can make the following ansatz $W = 0$ with:

$$W := K_e + vN + (v_0e_0 + v_1e_1 + v_2e_2 + v_3e_3)^2. \quad (17)$$

In the following we denote the coefficient of $e_0^je_1^ke_2^le_3^l$ of W by W_{ijkl} . We consider:

$$\begin{aligned} W_{1100} &= 2v_0v_1, & W_{1010} &= 2v_0v_2, \\ W_{0101} &= 2v_1v_3, & W_{0011} &= 2v_2v_3. \end{aligned} \quad (18)$$

which implies the following two cases:

- $v_0 = v_3 = 0$: We compute

$$W_{0200} - W_{2000} = v_1^2, \quad W_{0020} - W_{0002} = v_2^2, \quad (19)$$

which implies that all v_0, \dots, v_3 are equal to zero, a contradiction.

- $v_1 = v_2 = 0$: Now we compute

$$W_{2000} - W_{0200} = v_0^2, \quad W_{0002} - W_{0020} = v_3^2, \quad (20)$$

yielding the same contradiction.

This shows that the projection $K \rightarrow K_e$ cannot be birational, which is equivalent with the condition that the system of equations $S = \Delta_2 = \Delta_3 = \Delta_4 = \Delta_5 = 0$ linear in f_0, \dots, f_3 is linear dependent (rank of the coefficient matrix is less than 4). By means of linear algebra it can easily be verified that this can only be the case if $T = 0$ holds with

$$T := \varepsilon_{01}e_0e_1 + \varepsilon_{02}e_0e_2 + \varepsilon_{13}e_1e_3 + \varepsilon_{23}e_2e_3 \quad (21)$$

and

$$\varepsilon_{01} := \mu_3(1 + \mu_1)B, \quad \varepsilon_{02} := \mu_1A(\mu_3 + 1) - \mu_2B, \quad (22)$$

$$\varepsilon_{23} := \mu_3(1 - \mu_1)B, \quad \varepsilon_{13} := \mu_1A(\mu_3 - 1) + \mu_2B, \quad (23)$$

by using the following abbreviations:

$$A := A_5 - A_4 + 1, \quad B := B_4 - B_5. \quad (24)$$

Note that $T = 0$ is also a quadric in the Euler parameter space. In the general case $T = 0$ and $K_e = 0$ intersect along a curve, but due to Corollary 1 they have to possess at least a common 2-dimensional component (i.e. a plane) or they are even identical.

Necessary conditions for this circumstance are obtained by determining the intersection of $K_e = T = N = 0$ by resultant method; in detail this works as follows: We compute the resultant R_{K_e} of T and N with respect to e_0 . In the same way we compute the resultants R_T and R_N . Then we calculate the resultant R_{K_eT} of R_{K_e} and R_T with respect to e_3 . Analogously we obtain R_{K_eN} and R_{TN} . Finally we compute the greatest common divisor of R_{K_eT} , R_{K_eN} and R_{TN} , which can only vanish for $B_4B_5U_1U_2\mu_1\mu_3F_1F_2 = 0$ with

$$\begin{aligned} F_1 := & [B^2\mu_2 - AB(\mu_1 + \mu_3)](e_2^2 - e_1^2) + \\ & [2A^2\mu_1 - 2B^2\mu_3 - 2AB\mu_2]e_1e_2 \end{aligned} \quad (25)$$

and

$$F_2 := (1 + \mu_1)(\mu_3 - 1)e_1^2 + (1 + \mu_3)(\mu_1 - 1)e_2^2 - 2\mu_2 e_1 e_2. \quad (26)$$

In order that the three quadrics $N = T = K_e = 0$ have a curve (projected bonding curve) in common either F_1 or F_2 has to be fulfilled identically.

- ad F_1 : We solve the coefficient of e_2^2 of F_1 for μ_2 and plug the obtained expression in the coefficient of $e_1 e_2$ of F_1 . The resulting expression has only the real solution $A = B = 0$, a contradiction.
- ad F_2 : It can easily be seen that F_2 is fulfilled identically if and only if $\mu_1 = \mu_3 = 1$ and $\mu_2 = 0$ holds (due to our assumptions with respect to \mathbf{A} of Eq. (12)).

Therefore κ_2 has to be the identity, which already implies the geometric properties of the Duporcq pentapods. We only remain to show that these pentapods possess 2-dimensional self-motions, which are line-symmetric in addition.

4.3 Determining the pentapod's self-motion

Plugging $\mu_1 = \mu_3 = 1$ and $\mu_2 = 0$ in $T = 0$ yields: $-2e_0(Be_1 + Ae_2) = 0$. Therefore we have to distinguish two cases:

- $e_1 = -Ae_2/B$: Now K_e possesses 1397 terms and has to vanish identically as $\dim(K_e) = 2$ has to hold according to Corollary 1. We consider the coefficient of $e_0 e_3$ of K_e which equals $-8B_4 B_5 U_1 U_2$ and cannot vanish without contradiction.
- $e_0 = 0$: Now K_e factors into $(e_1^2 + e_2^2 + e_3^2)G[185]$ where G is of the form $g_0 + g_1 R_1^2 + g_2 R_2^2 + g_3 R_3^2 + g_4 R_4^2 + g_5 R_5^2$, where all g_i are functions in the geometry of the platform and the base. This equation can be solved for R_3^2 w.l.o.g. as g_3 equals $B^2 U_2^2 U_3$.

Now we compute f_0, f_1, f_3 from $S = \Delta_2 = \Delta_4 = 0$ w.l.o.g.. Plugging the obtained expressions into Δ_3 and Δ_5 imply in the numerators the following expressions:

$$-B_4 U_1 U_2 (e_1^2 + e_2^2 + e_3^2)H \quad \text{and} \quad (e_1^2 + e_2^2 + e_3^2)H, \quad (27)$$

respectively, with $H := h_1 e_1 + h_2 e_2$ and

$$\begin{aligned} h_1 &:= (R_2^2 - R_5^2)A_4 + (R_4^2 - R_1^2)(A_5 - 1), \\ h_2 &:= (R_2^2 - R_5^2)B_4 + (R_4^2 - R_1^2)B_5. \end{aligned} \quad (28)$$

As $\dim(K_e) = 2$ has to hold according to Corollary 1 the expression H has to be fulfilled identically. Under the assumption $R_1^2 \neq R_4^2$ we can compute A_5 and B_5 from $h_1 = h_2 = 0$ but then we get $U_3 = 0$; a contradiction. As a consequence $R_1^2 = R_4^2$ has to hold, which implies together with $h_1 = h_2 = 0$ the condition $R_2^2 = R_5^2$. This already yields a 2-dimensional self-motion. Moreover, due to $R_1^2 = R_4^2$ we get $f_0 = 0$, which proves the line-symmetry of this self-motion (cf. [3, Section 1]).

Remark 4. It should be noted that due to the existence of the similarity bond and $\kappa_2 = id$ the Duporcq pentapods also have pure translational 1-dimensional self-motions (cf. proof of Corollary 1). Moreover each 2-dimensional self-motion of a Duporcq pentapod contains a pure translational 1-dimensional sub-self-motion (obtained by $e_1 = e_2 = 0$). \diamond

5 Conclusions

In light of the results of this paper, Theorem 4 of [1] is not correct, but the flaw can be fixed by rewriting the phrase "which is not listed in Theorems 2 and 3" by "which is neither a Duporcq pentapod nor listed in Theorems 2 and 3".

Furthermore we have to check if the Duporcq pentapods do not imply a further case in the list of non-architecturally singular hexapods with 2-dimensional self-motions given in [1, Theorem 5]. Starting with a Duporcq pentapod, the two complete quadrilaterals are already determined and there is only one further point which has the same geometric properties with respect to this quadrilateral as the third anchor point. This is exactly the sixth anchor point illustrated in Fig. 1. But the resulting hexapod is architecturally singular as already mentioned in Subsection 1.1, thus Theorem 5 of [1] is correct.

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